Sciencia Acta Xaveriana
An International Science Journal
ISSN. 0976-1152

Volume 7
No. 2
pp. 81-110
September 2016

VERTEX-TO-EDGE MONOPHONIC DISTANCE IN GRAPHS

I. Keerthi Asir and S. Athisayanathan
Research Department of Mathematics
St. Xavier's College (Autonomous), Palayamkottai - 627 002
Tamilnadu, India.
e-mail : asirsxc@gmail.com, athisxc@gmail.com

Abstract

Let u be a vertex and e an edge in a connected graph G. A vertex-to-edge $u - e$ path P is a $u - v$ path, where v is a vertex in e such that P contains no vertices of e other than v and the $u - e$ path P is said to be $u - e$ monophonic path if P contains no chords in G. The vertex-to-edge monophonic distance $d_m(u, e)$ is the length of a longest $u - e$ monophonic path in G. A $u - e$ monophonic path of length $d_m(u, e)$ is called a vertex-to-edge $u - e$ monophonic. The vertex-to-edge monophonic eccentricity $e_{m_1}(u)$ of a vertex u in G is the maximum vertex-to-edge monophonic distance from u to an edge $e \in E$ in G. The vertex-to-edge monophonic radius R_{m_1} of G is the minimum vertex-to-edge monophonic eccentricity among the vertices of G, while the vertex-to-edge monophonic diameter D_{m_1} of G is the maximum vertex-to-edge monophonic eccentricity among the vertices of G. It is shown that $R_{m_1} \leq D_{m_1}$ for every connected graph G and that every two positive integers a and b with $2 \leq a \leq b$ are realizable as the vertex-to-edge monophonic radius and the vertex-to-edge monophonic diameter, respectively, of some connected graph. The vertex-to-edge monophonic center $C_{m_1}(G)$ is the subgraph induced by the set of all vertices having minimum vertex-to-edge monophonic eccentricity and the vertex-to-edge monophonic periphery $P_{m_1}(G)$ is the subgraph induced by the set of all vertices having maximum vertex-to-edge monophonic eccentricity. It is shown that the vertex-to-edge monophonic center of every connected graph G lies in a single block of G.

Keywords: vertex-to-edge distance, vertex-to-edge detour distance, vertex-to-edge monophonic distance.
Mathematics Subject Classification 2010: 05C12.

1 Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected simple graph. For basic graph theoretic terminologies, we refer to Chartrand and Zhang [2]. If $X \subseteq V$, then $\langle X \rangle$ is the subgraph induced by X. A Chord of a path $u_1, u_2, \ldots, u_n$ in G is an edge $u_i u_j$ with $j \geq i + 2$. For example if one is locating an emergency facility like police station, fire station, hospital, school, college, library, ambulance depot, emergency care center, etc., then the primary aim is to minimize the distance between the facility and the location of a possible emergency. In 1964 Hakimi [3] considered the facility location problems as vertex-to-vertex distance in graphs. For any two vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. For a vertex v in G, the eccentricity $e(v)$ of v is the distance between v and a vertex farthest from v in G. The minimum eccentricity among the vertices of G is its radius and the maximum eccentricity is its diameter, denoted by $rad(G)$ and $diam(G)$ respectively. A vertex v in G is a central vertex if $e(v) = rad(G)$ and the subgraph induced by the central vertices of G is the center $Cen(G)$ of G. A vertex v in G is a peripheral vertex if $e(v) = diam(G)$ and the subgraph induced by the peripheral vertices of G is the periphery $Per(G)$ of G. If every vertex of a graph G is a central vertex then G is called self-centered.

For example if one is making an election canvass or circular bus service the distance from the location is to be maximized. In 2005, Chartrand et. al. [1] introduced and studied the concepts of detour distance in graphs. For any two vertices u and v in a connected graph G, the detour distance $D(u, v)$ is the length of a longest $u - v$ path in G. For a vertex v in G, the detour eccentricity $e_D(v)$ of v is the detour distance between v and a vertex farthest from v in G. The minimum detour eccentricity among the vertices of G is its detour radius and the maximum detour eccentricity is its detour diameter, denoted by $rad_D(G)$ and $diam_D(G)$ respectively. The detour center, the detour self-centered and the detour periphery of a graph are defined similar to the center and periphery of a graph respectively.

For example if one is want to design the security based communication network, the monophonic concepts plays a vital role. These concepts have intresting applications in channel assignment problem in radio technologies. In 2011, Santhakumaran and Titus [7] introduced and studied the concepts of monophonic distance in graphs. For any two vertices u and v in G, a $u - v$ path P is a $u - v$ monophonic path if P contains no chords. The monophonic distance $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. For a vertex v in G, the monophonic eccentricity $e_m(v)$ of v is the monophonic distance between v and a vertex farthest from v in G. The minimum monophonic eccentricity among the vertices of G is its monophonic radius and the maximum monophonic eccentricity is its monophonic diameter, denoted by $rad_m(G)$ and $diam_m(G)$ respectively. The monophonic center, the monophonic self-centered and the monophonic periphery of a graph are defined similar to the center and periphery respectively of a graph.

For example when a railway line, pipe line or highway is constructed, the distance between the respective structure and each of the communities to be served is to be minimized. In a social network an edge represents two individuals having a common interest. Thus the vertex-to-edge distance have interesting application in social networks. In 2010, Santhakumaran [6] introduced the facility locational problem as vertex-to-edge distance in graphs as follows: For a vertex u and an edge e in a connected graph G, the vertex-to-edge distance is defined by $d(u,e) = min\{d(u,v) : v \in e\}$. The vertex-to-edge eccentricity $e_1(u)$ of u is defined by $e_1(u) = max\{d(u,e) : e \in E\}$. An edge e of G such that $e_1(u) = d(u,e)$ is called a vertex-to-edge eccentric edge of u. The vertex-to-edge radius r_1 of G is defined by $r_1 = min\{e_1(v) : v \in V\}$ and the vertex-to-edge diameter d_1 of G is defined by $d_1 = max\{e_1(v) : v \in V\}$. A vertex v for which $e_1(v)$ is minimum is called a vertex-to-edge central vertex of G and the set of all vertex-to-edge central vertices of G is the vertex-to-edge center $C_1(G)$ of G. A vertex v for which $e_1(v)$ is maximum is called a vertex-to-edge peripheral vertex of G and the set of all vertex-to-edge peripheral vertices of G is the vertex-to-edge periphery $P_1(G)$ of G. If every vertex of G is a vertex-to-edge central vertex then G is called vertex-to-edge self-centered graph.

Also when a dam, river or channel is constructed, the distance between the respective structure and each of the communities to be served is to be maximized. Keerthi Asir and Athisayanathan [4] introduced and studied the concepts of vertexto-edge detour distance in graphs as follows: For a vertex u and an edge e in a connected graph G, a vertex-to-edge $u - e$ path P is a $u - v$ path, where v is a vertex in e such that P contains no vertices of e other than v and the vertex-to-edge detour distance $D(u,e)$ is the length of a longest $u - e$ path. A $u - e$ path of length

$D(u,e)$ is called a vertex-to-edge $u-e$ detour and for our convenience a $u-e$ path of length $d(u,e)$ is called a vertex-to-edge $u-e$ geodesic or simply $u-e$ geodesic. The vertex-to-edge detour eccentricity, $e_{D_1}(u)$ of a vertex u in G is defined as $e_{D_1}(u) = max\ \{D(u,e) :\ e\ \in\ E\}$. An edge e for which $e_{D_1}(u) = D(u,e)$ is called a vertex-to-edge detour eccentric edge of u. The vertex-to-edge detour radius of G is defined by $R_1 = rad_{D_1}(G) = min\ \{e_{D_1}(v) : v \in V\}$ and the vertex-to-edge detour diameter of G is defined by $D_1 = diam_{D_1}(G) = max\ \{e_{D_1}(v) : v \in V\}$. A vertex v in G is called a vertex-to-edge detour central vertex if $e_{D_1}(v) = R_1$ and the vertex-to-edge detour center of G is defined by $C_{D_1}(G) = Cen_{D_1}(G) = \langle \{v \in V : e_{D_1}(v) = R_1\}\rangle$. A vertex v in G is called a vertex-to-edge detour peripheral vertex if $e_{D_1}(v) = D_1$ and the vertex-to-edge detour periphery of G is defined by $P_{D_1}(G) = Per_{D_1}(G) = \langle \{v \in V : e_{D_1}(v) = D_1\}\rangle$. If every vertex of G is a vertex-to-edge detour central vertex then G is called vertex-to-edge detour self-centered graph.

These motivated us to introduce a distance called the vertex-to-edge monophonic distance in graphs and investigate certain results related to vertex-to-edge monophonic distance and other distances in graphs. These ideas have interesting applications in channel assignment problem in radio technologies and capture different aspects of certain molecular problems in theoretical chemistry. Also there are useful applications of these concepts to security based communication network design. Throughout this paper, G denotes a connected graph with atleast two vertices.

2. Vertex-To-Edge Monophonic Distance

Definition 2.1. Let u be a vertex and e an edge in a connected graph G. A vertex-to-edge $u - e$ path P is said to be a vertex-to-edge $u - e$ monophonic path if P contains no chords. The vertex-to-edge monophonic distance, $d_m(u, e)$ is the length of a longest $u - e$ monophonic path in G. A $u - e$ monophonic path of length $d_m(u, e)$ is called a vertex-to-edge $u - e$ monophonic or simply $u - e$ monophonic.

Example 2.2. Consider the graph G given in Fig 2.1. For the vertex u and the edge $e = \{v, w\}$, the paths $P_1: u, w; P_2: u, z, r, v$ are $u - e$ monophonic paths, while the paths $Q_1: u, w, v; Q_2 : u, z, r, v, w; Q_3 : u, t, s, x, z, r, v$ and $Q_4 : u, t, s, x, y, z, r, v$ are not $u - e$ monophonic paths. Now the vertex-to-edge distance $d(u, e) = 1$, the vertex-to-edge detour distance $D(u, e) = 7$ and the vertex-to-edge monophonic distance $d_m(u, e) = 3$. Thus the vertex-to-edge monophonic distance is different from both the vertex-to-edge distance and the vertex-to-edge detour distance. Also P_1 is a $u - e$ geodesic, P_3 is a $u - e$ detour and P_2 is a u − e monophonic. Note that the $v - e$ and $w - e$ monophonic paths are trivial.

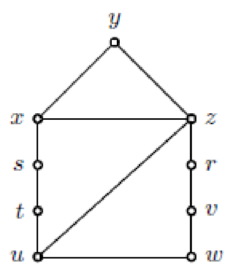

Fig. 2.1: G

Keerthi Asir and Athisayanathan [4] showed that for a vertex u and an edge e in a graph G of order n, $0 \leq d(u,e) \leq D(u,e) \leq n-2$. Now we have the following theorem.

Theorem 2.3. For a vertex u and an edge e in a non-trivial connected graph G of order n, $0 \leq d(u,e) \leq d_m(u,e) \leq D(u,e) \leq n-2$.

Proof. It is enough to show that $d(u,e) \leq d_m(u,e)$ and $d_m(u,e) \leq D(u,e)$. By definition $d(u,e) \leq d_m(u,e)$. If P is a unique u − e path in G, then $d(u,e) = dm(u,e) = D(u,e)$. Suppose that G contains more than one path. Let Q be a longest $u - e$ path in G.

***Case* 1.** If Q contains no chords in G, then $d_m(u,e) = D(u,e)$.

***Case* 2.** If Q contains a chord in G, then $dm(u,e) < D(u,e)$.

Remark 2.4. The bounds in the Theorem 2.3 are sharp. If G is a complete graph of order 2, then $d(u,e) = d_m(u,e) = D(u,e) = 0 = n-2$ for every vertex u in G and if G is a path $P: u_1, u_2, \ldots, u_{n-1}, u_n$ of order n, then $d(u,e) = d_m(u,e) = D(u,e) = n-2$, where $u = u_1$ and $e = \{u_{n-1}, u_n\}$. Also we note that if G is a tree, then $d(u,e) = d_m(u,e) = D(u,e)$ for every vertex u and edge e in G and the graph G given in Fig. 2.1, $0 < d(u,e) < d_m(u,e) < D(u,e) < n-2$, where $e = \{v,w\}$.

Since a vertex of degree $n-1$ in a graph G of order n, belongs to every edge e in G, we have the following theorem.

Theorem 2.5. Let G be a connected graph of order n and e an edge in G. If u is a vertex of degree $n-1$, then $d_m(u,e) = 0$.

The converse of the Theorem 2.5 is not true. For the graph G given in Fig. 2.1, $d_m(u,e) = 0$, where $e = \{u,w\}$, but $deg(u) \neq n - 1$.

Since the maximum length of a $u - e$ monophonic path between a vertex u and an edge e in $K_{n,m}$ is 2, we have the following theorem.

Theorem 2.6. Let $K_{n,m}(n < m)$ be a complete bipartite graph with the partition V_1, V_2 of $V(K_{n,m})$ such that $|V_1| = n$ and $|V_2| = m$. Let u be a vertex and e an edge such that $u \notin e$ in $K_{n,m}$, then $d_m(u,e) = 2$

Since every tree has unique $u - e$ monophonic path between a vertex u and an edge e, we have the following theorem.

Theorem 2.7. If G is a tree, then $d(u,e) = d_m(u,e) = D(u,e)$ for every vertex u and an edge e in G.

The converse of the Theorem 2.7 is not true. For every vertex u in K_3 with $u \notin e$,$d(u,e) = d_m(u,e) = D(u,e) = 1$ and for every vertex u in K_3 with $u \in e, d(u,e) = d_m(u,e) = D(u,e) = 0$.

3 Vertex-to-Edge Monophonic Center

Definition 3.1. Let G be a connected graph. The vertex-to-edge monophonic eccentricity $e_{m_1}(u)$ of a vertex u in G is defined as $e_{m_1}(u) = max\ \{d_m(u,e) : e \in E\}$. An edge e for which $e_{m_1}(u) = d_m(u,e)$ is called a vertex-to-edge monophonic eccentric edge of u. The vertex-to-edge monophonic radius of G is defined as,

$R_{m_1} = rad_{m_1}(G) = min\ \{e_{m_1}(v) : v \in V\}$ and the vertex-to-edge monophonic diameter of G is defined as, $D_{m_1} = diam_{m_1}(G) = max\ \{e_{m_1}(v) : v \in V\}$. A vertex v in G is called a vertex-to-edge monophonic central vertex if $e_{m_1}(v) = R_{m_1}$ and the vertex-to-edge monophonic center of G is defined as $C_{m_1}(G) = Cen_{m_1}(G) = \langle \{v \in V : e_{m_1}(v) = R_{m_1}\}\rangle$. A vertex v in G is called a vertex-to-edge monophonic peripheral vertex if $e_{m_1}(v) = D_{m_1}$ and the vertex-to-edge monophonic periphery of G is defined as, $P_{m_1}(G) = Per_{m_1}(G) = \langle \{v \in V : e_{m_1}(v) = D_{m_1}\}\rangle$. If every vertex of G is a vertex-to-edge monophonic central vertex, then G is called a vertex-to-edge monophonic self centered graph.

Example 3.2. For the connected graph G given in Fig. 3.1, the set of all edges in G are given by,

$E = \{e_1 = \{v_1, v_2\},\ e_2 = \{v_1, v_3\}, e_3 = \{v_2, v_3\},\ e_4 = \{v_3, v_4\},$
$e_5 = \{v_4, v_5\},\ e_6\{v_5, v_6\},\ e_7 = \{v_6, v_7\},\ e_8 = \{v_4, v_7\},\ e_9 =$
$\{v_7, v_8\}, e_{10} = \{v_8, v_{10}\},\ e_{11} = \{v_4, v_{10}\},\ e_{12} = \{v_4, v_9\},\ e_{13} =$
$\{v_9, v_{10}\},\ e_{14} = \{v_{10}, v_{14}\},\ e_{15} = \{v_{13}, v_{14}\},\ e_{16} = \{v_{12}, v_{13}\},$
$e_{17} = \{v_{11}, v_{12}\},\ e_{18} = \{v_{10}, v_{11}\},\ e_{19} = \{v_{11}, v_{14}\},\ e_{20} =$
$\{v_{10}, v_{11}\},\ e_{21} = \{v_{10}, v_{13}\},\ e_{22} = \{v_{10}, v_{12}\},\ e_{23} = \{v_{12}, v_{14}\}\}$.

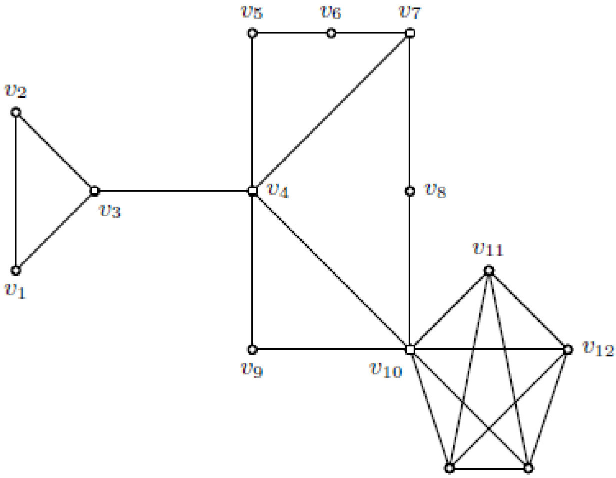

Fig. 3.1: G

The vertex-to-edge eccentricity $e_1(v)$ the vertex-to-edge detour eccentricity $e_{D_1}(v)$ and the vertex-to-edge monophonic eccentricity $e_{m_1}(v)$ of all the vertices of G are given in Table 1.

v	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}
$e_1(v)$	4	4	3	2	3	4	3	3	3	3	4	4	4	4
$e_{D_1}(v)$	11	11	9	8	8	8	8	9	9	7	11	11	11	11
$e_{m_1}(v)$	4	4	3	2	5	4	3	4	5	4	5	5	5	5

Table 1

The vertex-to-edge monophonic eccentric edge of all the vertices of G are given in Table 2.

Vertex v	*Vertex-to-Edge Monophonic Eccentric Edge e*
v_1, v_2, v_3	$e_6, e_7, e_9, e_{10}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{23}$
v_4	$e_9, e_{10}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{23}$
v_5	$e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{23}$
v_6, v_7	$e_1, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{23}$
v_8	e_1
$v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}$	e_5

Table 2

The vertex-to-edge radius $r_1 = 2$, the vertex-to-edge diameter $d_1 = 4$, the vertex-to-edge detour radius $R_1 = 7$, the vertex-to-edge detour diameter $D_1 = 11$, the vertex-to-edge monophonic radius $R_{m_1} = 2$ and the vertex-to-edge monophonic diameter $D_{m_1} = 5$. Also the vertex-to-edge center $C_1(G) = \langle\{v_4\}\rangle$, the vertex-to-edge periphery $P_1(G) = \langle\{v_1, v_2, v_6, v_{11}, v_{12}, v_{13}, v_{14}\}\rangle$, the vertex-to-edge detour center $C_{D_1}(G) = \langle\{v10\}\rangle$, the vertex-to-edge detour periphery $P_{D_1}(G) = \langle\{v_1, v_2, v_{11}, v_{12}, v_{13}, v_{14}\}\rangle$, the vertex-to-edge monophonic center $C_{m_1}(G) = \langle\{v_4\}\rangle$ and the vertex-to-edge monophonic periphery $P_{m_1}(G) = \{v_5, v_9, v_{11}, v_{12}, v_{13}, v_{14}\}$.

The vertex-to-edge monophonic radius R_{m_1} and the vertex-to-edge monophonic diameter D_{m_1} of some standard graphs are given in Table 3.

G	K_n	P_n	$C_n(n \geq 4)$	$W_n(n \geq 5)$	$K_{n,m}(m \geq n)$
R_{m_1}	1	$\lfloor \frac{n-2}{2} \rfloor$	$n-2$	1	2
D_{m_1}	1	$n-2$	$n-2$	$n-3$	2

Table 3

Remark 3.3. In a connected graph $G, C_1(G), C_{D_1}(G), C_{m_1}(G)$ and $P_1(G), P_{D_1}(G), P_{m_1}(G)$ need not be same. For the graph G given in Fig 3.2, it is shown that $C_1(G) = \langle\{v_4, v_9\}\rangle, C_{D_1}(G) = \langle\{v_{10}\}\rangle$ and $C_{m_1}(G) = \langle\{v_6\}\rangle$ are distinct. Also the graph G given in Fig 3.1, it is shown that

$P_1(G) = \langle\{v_1, v_2, v_6, v_{11}, v_{12}, v_{13}, v_{14}\}\rangle$,

$P_{D_1}(G) = \langle\{v_1, v_2, v_{11}, v_{12}, v_{13}, v_{14}\}\rangle$

$P_{m_1}(G) = \{v_5, v_9, v_{11}, v_{12}, v_{13}, v_{14}\}$

are distinct.

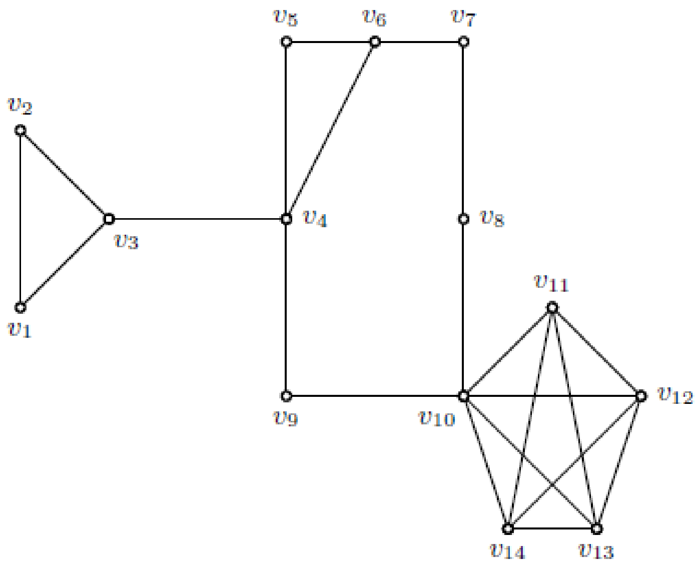

Fig. 3.2: G

Remark 3.4. In a connected graph $G, C_{m_1}(G)$ and $P_{m_1}(G)$ need not be connected. For the graph G given in Fig 3.3, $C_{m_1}(G) = \langle\{v_2, v_4\}\rangle$ and $P_{m_1}(G) = \langle\{v_1, v_3, v_5\}\rangle$ are disconnected.

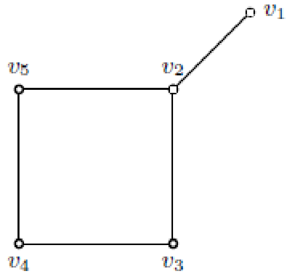

Fig. 3.3: G

Example 3.5. The complete graph K_n, the cycle C_n and the complete bipartite graph $K_{n,m}$ are vertex-to-edge monophonic self centered graphs.

Remark 3.6. A self-centered graph need not be a vertex-to-edge monophonic self centered graph. For the graph G given in Fig 3.4, $C(G) = \langle V(G) \rangle$ and $C_{m_1}(G) = \langle \{v_2, v_4, v_7\} \rangle$.

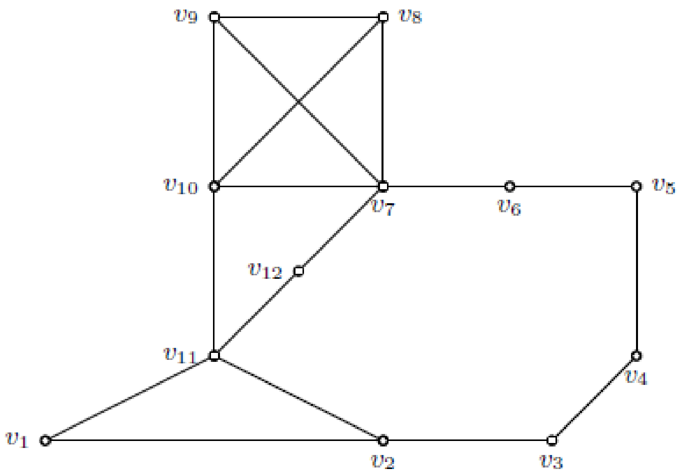

Fig. 3.4: G

Remark 3.7. A vertex-to-edge self-centered graph need not be a vertex-to-edge monophonic self centered graph. For the graph G given in Fig 3.5, $C_1(G) = \langle V\ (G)\rangle$ and $C_{m_1}(G) = \langle\{v_2, v_6, v_8\}\rangle$.

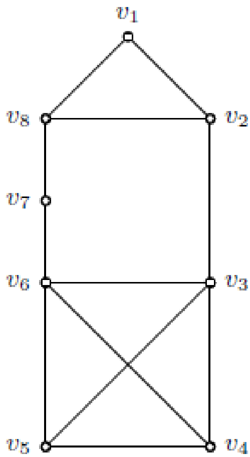

Fig. 3.5: G

Remark 3.8. A detour self-centered graph need not be a vertex-to-edge monophonic self centered graph. For the wheel $W_n = K_1 + C_{n-1}(n \geq 5)$, $C_D(Wn) = \langle V\ (W_n)\rangle$ and $C_{m_1}(Wn) = \langle V\ (K1)\rangle$.

Remark 3.9. A vertex-to-edge detour self-centered graph need not be a vertex-to-edge monophonic self centered graph. For the graph G given in Fig 3.6, $C_{D_1}(G) = \langle V\ (G)\rangle$ and $C_{m_1}(G) = \langle\{v_1, v_3\}\rangle$.

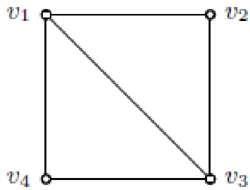

Fig. 3.6: G

Observation 3.10. No cut vertex in a connected graph G is a vertex-to-edge monophonic peripheral vertex of G.

Remark 3.11. If G is a vertex-to-edge monophonic self-centered graph, then G is its own vertex-to-edge monophonic periphery.

Theorem 3.12. Let G be a connected graph of order n. Then

(i) $0 \leq e_1(v) \leq e_{m_1}(v) \leq e_{D_1}(v) \leq n-2$ for any vertex v in G.

(ii) $0 \leq r_1 \leq R_{m_1} \leq R_1 \leq n-2$.

(iii) $0 \leq d_1 \leq D_{m_1} \leq D_1 \leq n-2$.

Proof. This follows from Theorem 2.3.

Remark 3.13. The bounds in the Theorem 3.12(i) are sharp. If G is a complete graph of order n, then $e_1(u) = e_{m_1}(u) = e_{D_1}(u) = 0$ for every vertex u in G and if G is a path $P: u_1, u_2, \ldots, u_{n-1}, u_n$ of order n, then $e_1(u) = e_{m_1}(u) = e_{D_1}(u) = n-2$, where $u = u_1$ or $u = u_n$. Also we note that if G is a tree, then $e_1(u) = e_{m_1}(u) = e_{D_1}(u)$ for every vertex u in G and for the graph G given in Fig. 2.1, $e_1(u) < e_{m_1}(u) < e_{D_1}(u)$.

In [1, 2] it is shown that for every non-trivial connected graph G, the radius and diameter are related by $rad(G) \leq diam(G) \leq 2\,rad(G)$, the detour radius and detour diamater are related $by\,rad_D(G) \leq diam_D(G) \leq 2\,rad_D(G)$ and Keerthi Asir et. al. [4] showed that the vertex-to-edge detour radius and vertex-to-edge detour diameter are related by $R_1 \leq D_1 \leq 2R_1 + 1$. The following example shows that the upper inequality does not hold for the vertex-to-edge monophonic distance also.

Example 3.14. For the wheel

$$W_n(n \geq 5), D_{m_1} > 2R_{m_1} \text{ and } D_{m_1} > 2R_{m_1} + 1.$$

Ostrand [5] showed that for each pair a, b of positive integers with $a \leq b \leq 2a$ are realizable as the radius and diameter respectively of some connected graph, Chartrand et. al. [1] showed that for each pair a, b of positive integers with $a \leq b \leq 2a$ are realizable as the detour radius and detour diameter respectively of some connected graph and Keerthi Asir et. al. [4] showed that every two positive positive integers a and b with $a \leq b \leq 2a + 1$ are realizable as the vertex-to-edge detour radius and vertex-to-edge detour diameter respectively of some connected graph. Now, we have the following realization theorem for the vertex-to-edge monophonic radius and vertex-to-edge monophonic diameter of some connected graph.

Theorem 3.15. For each pair a, b of positive integers with $2 \leq a \leq b$, there exists a connected graph G with $R_{m_1} = a$ and $D_{m_1} = b$.

Proof. ***Case 1***. $a = b$. Let $G = C_{a+2} : u_1, u_2, \ldots, u_{a+2}, u_1$ be a cycle of order $a + 2$. Then $e_{m_1}(u_i) = a$ for $1 \leq i \leq a + 2$. It is easy to verify that every vertex x in G with $e_{m_1}(x) = a$. Thus $R_{m_1} = a$ and $D_{m_1} = b$ as $a = b$.

Case 2. $2 \le a < b \le 2a$. Let $C_{a+2} : u_1, u_2, \ldots, u_{a+2}, u_1$ be a cycle of order $a + 2$ and $P_{b-a+1} : v_1, v_2, \ldots, v_{b-a+1}$ be a path of order $b - a + 1$. We construct the graph G of order $b + 2$ by identifying the vertex u_1 of C_{a+2} with v_1 of P_{b-a+1} as shown in the Fig 3.7. It is easy to verify that

$$e_{m_1}(u_i) = a \text{ for } i = 1, 2, a+2$$

$$e_{m_1}(u_i) = \begin{cases} b - i + 2, & \text{if } 3 \le i \le \lceil \frac{a+2}{2} \rceil \\ b - a + i - 2, & \text{if } \lceil \frac{a+2}{2} \rceil < i \le a+1 \end{cases}$$

$$e_{m_1}(v_i) = a + i - 1 \text{ for } 1 \le i \le b - a + 1$$

In particular

$$e_{m_1}(u_i) = b \text{ for } i = 3, a+1$$

$$e_{m_1}(v_i) = b \text{ for } i = b - a + 1$$

It is easy to verify that there is no vertex x in G with $e_{m_1}(x) < a$ and there is no vertex y in G with $e_{m_1}(y) > b$. Thus $R_{m_1} = a$ and $D_{m_1} = b$ as $a < b$.

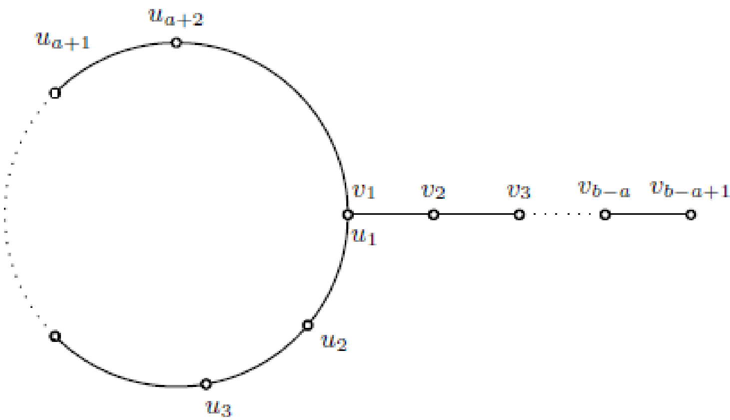

Fig. 3.7: G

***Case* 3.** $b > 2a$. Let G be a graph of order $b + 2a + 3$ obtained by identifying the central vertex of the wheel $W_{b+3} = K_1 + C_{b+2}$ and an end vertex of the path P_{2a+1}, where $K_1 : v_1$, $C_{b+2} : u_1, u_2, \ldots, u_{b+2}, u_1$ and $P_{2a+1} : v_1, v_2, \ldots, v_{2a+1}$. The resulting graph G is shown in Fig. 3.8.

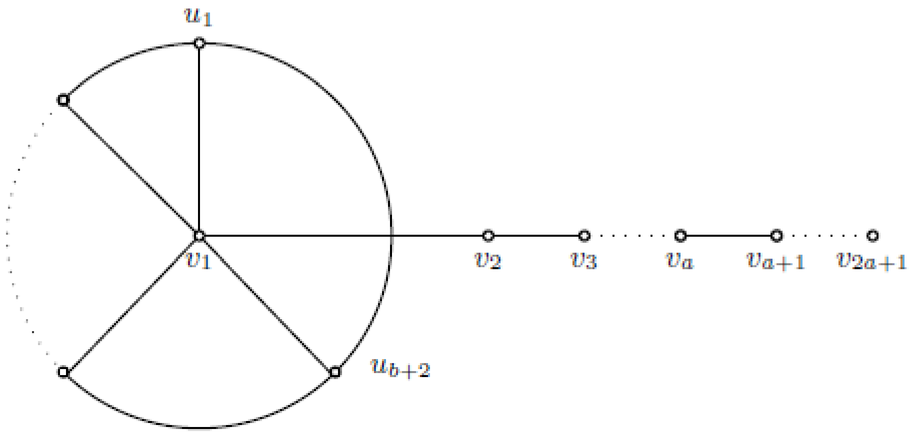

Fig. 3.8: G

It is easy to verify that

$$e_{m_1}(u_i) = b \text{ for } 1 \leq i \leq b+2$$

$$e_{m_1}(v_i) = \begin{cases} 2a - i, & \text{if } 1 \leq i \leq a \\ i - 1, & \text{if } a < i \leq 2a+1 \end{cases}$$

It is easy to verify that there is no vertex x in G with $e_{m_1}(x) < a$ and there is no vertex y in G with $e_{m_1}(y) > b$. Thus $R_{m_1} = a$ and $D_{m_1} = b$ as $b > 2a$.

Now we have a realization theorem for the vertex-to-edge radius, vertex-to-edge monophonic radius and vertex-to-edge detour radius of some connected graph.

Theorem 3.16. For any three positive integers a, b, c with $3 \leq a \leq b \leq c$, there exists a connected graph G such that $r_1 = a, R_{m_1} = b$ and $R_1 = c$.

Proof. ***Case 1.*** $a = b = c$. Let $P_1 : u_1, u_2, \ldots, u_{a+1}$ and $P_2 : v_1, v_2, \ldots, v_{a+1}$ be two paths of order $a + 1$. We construct the graph G of order $2a + 2$ by joining u_1 of P_1 and v_1 of P_2 by an edge. It is easy to verify that $e_1(u_i) = e_{m_1}(u_i) = e_{D_1}(u_i) = a + i - 1$ for $1 \leq i \leq a + 1$. Also $e_1(v_i) = e_{m_1}(v_i) = e_{D_1}(v_i) = a + i - 1$ for $1 \leq i \leq a + 1$. It is easy to verify that there is no vertex x in G with $e_1(x) < a, e_{m_1}(x) < b$ and $e_{D_1}(x) < c$. Thus $r_1 = a, R_{m_1} = b$ and $R_1 = c$. as $a = b = c$.

Case 2. $3 \leq a \leq b < c$. Let $F_1 : u_1, u_2, \ldots, u_{a+1}$ and $F_2 : v_1, v_2, \ldots, v_{a+1}$ be two paths of order $a + 1$. Let $F_3 : w_1, w_2, \ldots, w_{b-a+3}$ and $F_4 : z_1, z_2, \ldots, z_{b-a+3}$ be two paths of order $b - a + 3$. Let $F_5 = K_{c-b+1}$ with $V(F_5) = \{x_1, x_2, \ldots, x_{c-b+1}\}$. We construct the graph G of order $c + b + 3$ as follows: (i) identify the vertices u_1 in F_1, w_1 in F_3 and x_1 in F_5; also identify the vertices v_1 in F_2, z_1 in F_4 and x_{c-b+1} in F_5 (ii) identify the vertices u_3 in F_1 and w_{b-a+3} in F_3; also and identify the vertices z_{b-a+3} in F_4 and v_3 in F_2, (iii) join each vertex $w_i (2 \leq i \leq b - a + 2)$ in F_3 and u_2 in F_1 and join each vertex $z_i (2 \leq i \leq b - a + 2)$ in F_4 and v_2 in F_2. The resulting graph G is shown in Fig. 3.9.

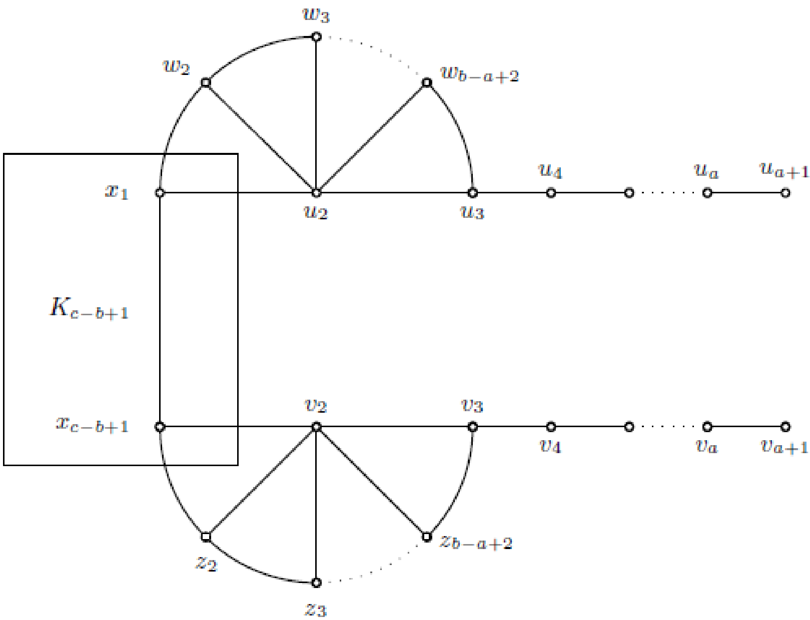

Fig. 3.9: G

It is easy to verify that

$e_1(x_i) = a$ for $1 \le i \le c-b+1$
$e_1(u_i) = a+i-1$ for $1 \le i \le a+1$
$e_1(v_i) = a+i-1$ for $1 \le i \le a+1$

$$e_1(w_i) = \begin{cases} a+i-1, & \text{if } 1 \le i \le 2 \\ a+2, & \text{if } 3 \le i \le b-a+3 \end{cases}$$

$$e_1(z_i) = \begin{cases} a+i-1, & \text{if } 1 \le i \le 2 \\ a+2, & \text{if } 3 \le i \le b-a+3 \end{cases}$$

$e_{m_1}(x_i) = b$ for $1 \le i \le c-b+1$

$$e_{m_1}(u_i) = \begin{cases} b+i-1, & \text{if } 1 \le i \le 2 \\ 2b-a+i-1, & \text{if } 3 \le i \le a+1 \end{cases}$$

$$e_{m_1}(v_i) = \begin{cases} b+i-1, & \text{if } 1 \le i \le 2 \\ 2b-a+i-1, & \text{if } 3 \le i \le a+1 \end{cases}$$

$e_{m_1}(w_i) = b+i-1$ for $1 \le i \le b-a+3$
$e_{m_1}(z_i) = b+i-1$ for $1 \le i \le b-a+3$

$$e_{D_1}(u_i) = \begin{cases} c, & \text{if } i = 1 \\ c+b-a+3, & \text{if } i = 2 \\ c+b-a+i, & \text{if } 3 \le i \le a+1 \end{cases}$$

$e_{D_1}(x_i) = c$ for $1 \le i \le c-b+1$

$$e_{D_1}(v_i) = \begin{cases} c, & \text{if } i = 1 \\ c+b-a+3, & \text{if } i = 2 \\ c+b-a+i, & \text{if } 3 \le i \le a+1 \end{cases}$$

$$e_{D_1}(w_i) = \begin{cases} c, & \text{if } i = 1 \\ c+b-a+3, & \text{if } 2 \le i \le b-a+3 \end{cases}$$

$$e_{D_1}(z_i) = \begin{cases} c, & \text{if } i = 1 \\ c+b-a+3, & \text{if } 2 \le i \le b-a+3 \end{cases}$$

It is easy to verify that there is no vertex x in G with $e_1(x) < a, e_{m_1}(x) < b$ and $e_{D_1}(x) < c$. Thus $r_1 = a, R_{m_1} = b$ and $R_1 = c$ as $a \le b < c$.

Case 3. $3 \leq a < b = c$. Let $E_1 : v_1, v_2, \ldots, v_{2a+3}$ be a path of order $2a + 3$. Let $E_2 : u_1, u_2, \ldots, u_{b-a+3}$ and $E_3 : w_1, w_2, \ldots, w_{b-a+3}$ be two paths of order $b - a + 3$. Let $E_i : z_i (4 \leq i \leq 2(b - a) + 3)$ be $2(b - a)$ copies of K_1. We construct the graph G of order $4b - 2a + 5$ as follows: (i) identify the vertices v_{a+2} in E_1; u_1 in E_2 and w_1 in E_3 (ii) identify the vertices v_a in E_1 and u_{b-a+3} in E_2 and identify the vertices v_{a+4} in E_1 and w_{b-a+3} in E_3 (iii) join each $E_i (4 \leq i \leq b - a + 3)$ with v_{a+2} in E_1 and w_{i-1} in E_2 and join each $E_i (b - a + 4 \leq i \leq 2(b - a) + 3)$ with v_{a+2} in E_1 and $u_{i-b+a-1}$ in E_3. The resulting graph G is shown in Fig. 3.10.

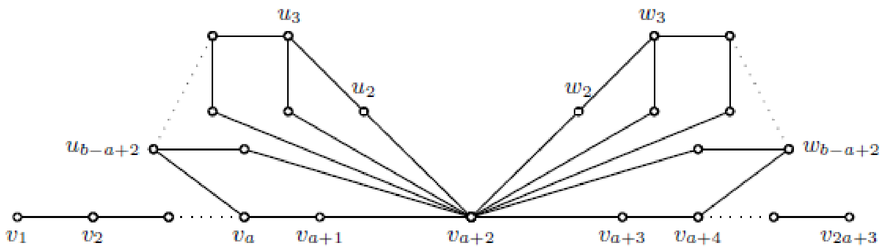

Fig. 3.10: G

It is easy to verify that

$$e_1(v_i) = \begin{cases} 2a+2-i, & \text{if } 1 \le i \le a+1 \\ a, & \text{if } i = a+2 \\ i-2, & \text{if } a+3 \le i \le 2a+3 \end{cases}$$

$$e_1(w_i) = \begin{cases} a, & \text{if } i = 1 \\ a+1, & \text{if } i = 2 \\ a+2, & \text{if } 3 \le i \le b-a+3 \end{cases}$$

$$e_1(u_i) = \begin{cases} a, & \text{if } i = 1 \\ a+1, & \text{if } i = 2 \\ a+2, & \text{if } 3 \le i \le b-a+3 \end{cases}$$

$$e_1(z_i) = a+1 \text{ for } 4 \le i \le 2(b-a)+3$$

$$e_{m_1}(v_i) = \begin{cases} 2b+2-i, & \text{if } 1 \le i \le a \\ b+1, & \text{if } i = a+1, a+3 \\ b, & \text{if } i = a+2 \\ 2(b-a)-2+i, & \text{if } a+4 \le i \le 2a+3 \end{cases}$$

$$e_{m_1}(w_i) = \begin{cases} b+i-1, & \text{if } i = 1,2 \\ 2b-a+5-i, & \text{if } 3 \le i \le \left\lfloor \frac{b-a+5}{2} \right\rfloor \\ b+i-1, & \text{if } \left\lfloor \frac{b-a+5}{2} \right\rfloor < i \le b-a+3 \end{cases}$$

$$e_{m_1}(u_i) = \begin{cases} b+i-1, & \text{if } i = 1,2 \\ 2b-a+5-i, & \text{if } 3 \le i \le \left\lfloor \frac{b-a+5}{2} \right\rfloor \\ b+i-1, & \text{if } \left\lfloor \frac{b-a+5}{2} \right\rfloor < i \le b-a+3 \end{cases}$$

$$e_{m_1}(z_i) = b+1 \text{ for } 4 \le i \le 2(b-a)+3$$

$$e_{D_1}(v_i) = \begin{cases} 2b+2-i, & \text{if } 1 \le i \le a \\ 2b-a+3, & \text{if } i = a+1, a+3 \\ b, & \text{if } i = a+2 \\ 2(b-a)-2+i, & \text{if } a+4 \le i \le 2a+3 \end{cases}$$

$$e_{D_1}(w_i) = \begin{cases} b, & \text{if } i = 1 \\ 2b-a+5-i, & \text{if } 2 \le i \le \left\lfloor \frac{b-a+5}{2} \right\rfloor \\ b+i-1, & \text{if } \left\lfloor \frac{b-a+5}{2} \right\rfloor < i \le b-a+3 \end{cases}$$

$$e_{D_1}(u_i) = \begin{cases} b, & \text{if } i = 1 \\ 2b - a + 5 - i, & \text{if } 2 \le i \le \left\lfloor \frac{b-a+5}{2} \right\rfloor \\ b + i - 1, & \text{if } \left\lfloor \frac{b-a+5}{2} \right\rfloor < i \le b - a + 3 \end{cases}$$

$$e_{D_1}(z_i) = \begin{cases} e_{D_1}(w_{i-2}), & \text{if } 4 \le i \le \left\lfloor \frac{b-a+7}{2} \right\rfloor \\ e_{D_1}(w_i), & \text{if } \left\lfloor \frac{b-a+7}{2} \right\rfloor < i \le b - a + 3 \\ e_{D_1}(u_{i-b+a-2}), & \text{if } 4 \le i \le \left\lfloor \frac{b-a+7}{2} \right\rfloor \\ e_{D_1}(u_{i-b+a}), & \text{if } \left\lfloor \frac{b-a+7}{2} \right\rfloor < i \le b - a + 3 \end{cases}$$

It is easy to verify that there is no vertex x in G with $e_1(x) < a, e_{m_1}(x) < b$ and $e_{D_1}(x) < c$. Thus $r_1 = a, R_{m_1} = b$ and $R_1 = c$ as $a < b = c$.

Now we have a realization theorem for the vertex-to-edge diameter, the vertex-to-edge monophonic diameter and the vertex-to-edge detour diameter of some connected graph.

Theorem 3.17. For any three positive integers a, b and c with $4 \le a \le b \le c$, there exists a connected graph G such that $d_1 = a, D_{m_1} = b$ and $D_1 = c$.

Proof. *Case* 1. $a = b = c$. Let $G = P_{a+2} : u_1, u_2, \ldots, u_{a+2}$ be a path of order $a + 2$. Then

$$e_1(u_i) = e_{m_1}(u_i) = e_{D_1}(u_i) = \begin{cases} a + 1 - i, & \text{if } 1 \le i \le \left\lfloor \frac{a+2}{2} \right\rfloor \\ i - 2, & \text{if } \left\lfloor \frac{a+2}{2} \right\rfloor < i \le a + 2 \end{cases}$$

It is easy to verify that there is no vertex x in G with $e_1(x) > a, e_{m_1}(x) > b$ and $e_{D_1}(x) > c$. Thus $d_1 = a, D_{m_1} = b$ and $D_1 = c$ as $a = b = c$.

***Case* 2.** $4 \le a \le b < c$. Let $F_1 : u_1, u_2, \ldots, u_{a+1}$ be a path of order $a + 1$. Let $F_2 : w_1, w_2, \ldots, w_{b-a+3}$ be a path of order $b - a + 3$. Let

$F_3: x_1, x_2, \ldots, x_{c-b+1}$ be a complete graph of order $c - b + 1$. We construct the graph G of order $c + 2$ as follows: (i) identify the vertices u_1 in F_1; w_1 in F_2 and x_1 in F_3 and identify the vertices u_3 in F_1 and w_{b-a+3} in F_2, (ii) join each vertex $w_i (2 \le i \le b - a + 2)$ in F_2 and u_2 in F_1. The resulting graph G is shown in Fig. 3.11.

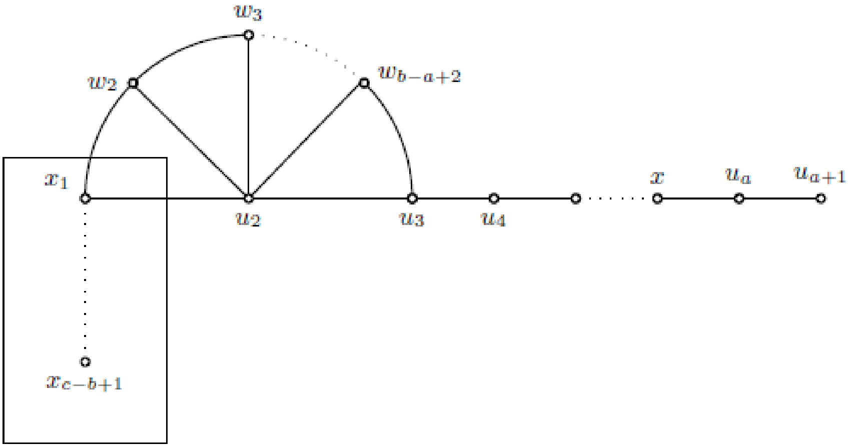

Fig. 3.11: G

It is easy to verify that

$$e_1(x_i) = \begin{cases} a-1, & \text{if } i = 1 \\ a, & \text{if } 2 \le i \le c - b + 1 \end{cases}$$

$$e_1(u_i) = \begin{cases} a-i, & \text{if } 1 \le i \le \left\lfloor \frac{a+1}{2} \right\rfloor \\ i-1, & \text{if } \left\lfloor \frac{a+1}{2} \right\rfloor < i \le a+1 \end{cases}$$

$$e_1(w_i) = \begin{cases} a-1, & \text{if } 1 \le i \le b - a + 1 \\ a-2, & \text{if } i = b - a + 2 \\ a-3, & \text{if } i = b - a + 3 \text{ for } a - 3 \ge 2 \\ 2, & \text{if } i = b - a + 3 \text{ for } a - 3 \le 2 \end{cases}$$

$$e_{m_1}(u_i) = \begin{cases} b-1, & \text{if } i=1 \\ a-2, & \text{if } i=2 \\ b-a+i-1, & \text{if } 3 \le i \le a+1 \text{ for } b-a+i-1 \ge a-i \\ a-i, & \text{if } 3 \le i \le a+1 \text{ for } b-a+i-1 \le a-i \end{cases}$$

$$e_{m_1}(x_i) = \begin{cases} b-1, & \text{if } i=1 \\ b, & \text{if } 2 \le i \le c-b+1 \end{cases}$$

$$e_{m_1}(w_i) = \begin{cases} b-i, & \text{if } 1 \le i \le \lfloor \frac{b}{2} \rfloor \text{ for } \lfloor \frac{b}{2} \rfloor < b-a+3 \\ i-1, & \text{if } \lfloor \frac{b}{2} \rfloor < i \le b-a+3 \text{ for } \lfloor \frac{b}{2} \rfloor < b-a+3 \\ b-i, & \text{if } 1 \le i \le b-a+3 \text{ for } \lfloor \frac{b}{2} \rfloor > b-a+3 \end{cases}$$

$$e_{D_1}(x_i) = \begin{cases} b, & \text{if } i=1 \\ c, & \text{if } 2 \le i \le c-b+1 \end{cases}$$

$$e_{D_1}(u_i) = \begin{cases} b, & \text{if } i=1,2 \\ b-a+i, & \text{if } 3 \le i \le a+1 \text{ for } b-a+i \ge a-i \\ a-i, & \text{if } 3 \le i \le a+1 \text{ for } b-a+i \le a-i \end{cases}$$

$$e_{D_1}(w_i) = \begin{cases} b, & \text{if } 1 \le i \le b-a+2 \\ b-a+3, & \text{if } i=b-a+3 \text{ for } b-a+3 \ge a-3 \\ a-3, & \text{if } i=b-a+3 \text{ for } b-a+3 \le a-3 \end{cases}$$

It is easy to verify that there is no vertex x in G with $e_1(x) > a, e_{m_1}(x) > b$ and $e_{D_1}(x) > c$. Thus $d_1 = a, D_{m_1} = b$ and $D_1 = c$ as $a \le b < c$.

Case 3. $4 \le a < b = c$. Let $E_1 : v_1, v_2, \ldots, v_{a+2}$ be a path of order $a+2$. Let $E_2 : w_1, w_2, \ldots, w_{b-a+3}$ be a path of order $b-a+3$. Let $E_i : z_i (3 \le i \le b-a+2)$ be $b-a$ copies of K_1. We construct the graph G of order $2b-a+3$ as follows: (i) identifying the vertices v_{a-1} and v_{a+1} of E_1 with w_1 and w_{b-a+3} of E_2 respectively and joining each $z_i (3 \le i \le b-a+2)$ with v_{a-1} in E_1 and w_i in E_2. The resulting graph G is shown in Fig. 3.12.

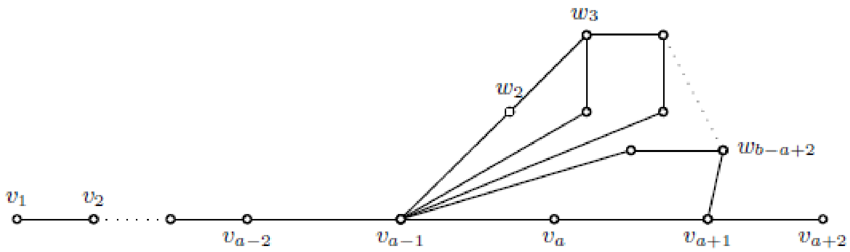

Fig. 3.12: G

It is easy to verify that

$$e_1(v_i) = \begin{cases} a+1-i, & \text{if } 1 \leq i \leq \lfloor \frac{a+2}{2} \rfloor \\ i-2, & \text{if } \lfloor \frac{a+2}{2} \rfloor < i \leq a+2 \end{cases}$$

$$e_{m_1}(v_i) = \begin{cases} b+1-i, & \text{if } 1 \leq i \leq a-1 \\ b-1, & \text{if } i = a+1 \\ b-a+2, & \text{if } i = a \text{ for } b-a+2 \geq a-2 \\ a-2, & \text{if } i = a \text{ for } b-a+2 < a-2 \\ b, & \text{if } i = a+2 \end{cases}$$

$$e_{D_1}(v_i) = \begin{cases} b+1-i, & \text{if } 1 \leq i \leq a-1 \\ b-1, & \text{if } i = a+1 \\ b, & \text{if } i = a, a+2 \end{cases}$$

$$e_1(w_i) = \begin{cases} e_1(v_{a-1}), & \text{if } i = 1 \\ 2, & \text{if } i = 2 \text{ for } a = 4 \text{ and } b-a = 1 \\ 3, & \text{if } i = 2 \text{ for } a = 4 \text{ and } b-a > 1 \\ a-2, & \text{if } i = 2 \text{ for } a > 4 \text{ and } b-a > 1 \\ a-1, & \text{if } 3 \leq i \leq b-a+3 \end{cases}$$

$$e_{m_1}(w_i) = \begin{cases} e_{m_1}(v_{a-1}), & \text{if } i = 1 \\ b - a + 2, & \text{if } i = 2 \text{ for } b - a + 2 \geq a - 2 \\ a - 2, & \text{if } i = 2 \text{ for } b - a + 2 < a - 2 \\ b + 2 - i, & \text{if } 3 \leq i \leq \lceil \frac{b-a+5}{2} \rceil \\ a - 4 + i, & \text{if } \lceil \frac{b-a+5}{2} \rceil < i \leq b - a + 3 \end{cases}$$

$$\{ e_{D_1}(v_{a-1}), \quad \text{if } i = 1$$

$$e_{D_1}(w_i) = \begin{cases} e_{D_1}(v_{a-1}), & \text{if } i = 1 \\ b + 2 - i, & \text{if } 2 \leq i \leq \lceil \frac{b-a+5}{2} \rceil \\ a - 4 + i, & \text{if } \lceil \frac{b-a+5}{2} \rceil < i \leq b - a + 3 \end{cases}$$

$$e_1(z_i) = e_1(w_2) \text{ if } 3 \leq i \leq b - a + 2$$

$$e_{m_1}(z_i) = \begin{cases} b - a + 5 - i, & \text{if } 3 \leq i \leq b - a + 2 \text{ for } b - a + 5 - i \geq i - 1 \\ i - 1, & \text{if } 3 \leq i \leq b - a + 2 \text{ for } b - a + 5 - i < i - 1 \end{cases}$$

$$e_{D_1}(z_i) = e_{D_1}(w_i) + 1 \text{ if } 3 \leq i \leq b - a + 2$$

It is easy to verify that there is no vertex x in G with $e_1(x) > a, e_{m_1}(x) > b$ and $e_{D_1}(x) > c$. Thus $d_1 = a, D_{m_1} = b$ and $D_1 = c$ as $a < b = c$.

Theorem 3.18. The vertex-to-edge monophonic center of every connected graph G lies in a single block of G.

Proof. Suppose that the vertex-to-edge monophonic center of a connected graph G lies in more than one block. Then G contains a cut vertex v such that $G - v$ has two components G_1 and G_2, each of which contains a vertex-to-edge monophonic central vertices of G. Let C be a vertex-to-edge monophonic eccentric edge of v and let P be a longest vertex-to-edge monophonic path in G. At least one of G_1 and G_2 contains no vertices of P, say G_2 contains no vertex of P. Let w be a vertex-to-edge monophonic central vertex in G that belongs to G_2 and let Q be a longest $w - v$ monophonic path in G. Since v is a cut vertex, P followed by Q produces a longest $w - e$ monophonic path, whose length is greater than that of P. Hence

$$e_{m_1}(w) \geq d_m(w,v) + d_m(v,e) = d_m(w,v) + e_{m_1}(v) > e_{m_1}(v)$$

Thus $e_{m_1}(w) > e_{m_1}(v)$. So that w is not a vertex-to-edge monophonic central vertex in G, which is contradiction. Hence $C_{m_1}(G)$ lies within a block of G.

Corollary 3.19. The vertex-to-edge monophonic center of every tree is isomorphic to either K_1 or K_2.

Proof. Every edges of a tree is a block. So that the vertex-to-edge monophonic center of every tree is either a vertex or the adjacent vertices.

Now we leave the following problems as open.

Problem 3.20. Does there exists a connected graph G such that $e_1(v) \neq e_{m_1}(v) \neq e_{D_1}(v)$ for every vertex v in G?

Problem 3.21. Is every graph a vertex-to-edge monophonic center of some connected graph?

Problem 3.22. Is every graph a vertex-to-edge monophonic periphery of some connected graph?

Problem 3.23. Characterize the vertex-to-edge monophonic self-centered graphs.

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